

# DISCONTINUITY OF THE LEMPERT FUNCTION AND THE KOBAYASHI-ROYDEN METRIC OF THE SPECTRAL BALL

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ABSTRACT. Some results on the discontinuity properties of the Lempert function and the Kobayashi pseudometric in the spectral ball are given.

## 1. INTRODUCTION AND RESULTS

Let  $\mathcal{M}_n$  be the set of all  $n \times n$  complex matrices. For  $A \in \mathcal{M}_n$  denote by  $sp(A)$  and  $r(A) = \max_{\lambda \in sp(A)} |\lambda|$  the spectra and the spectral radius of  $A$ , respectively. The spectral ball  $\Omega_n$  is the set

$$\Omega_n = \{A \in \mathcal{M}_n : r(A) < 1\}.$$

The Nevanlinna–Pick problem in  $\Omega_n$  (or the spectral Nevanlinna–Pick problem) is the following one: given  $N$  points  $a_1, \dots, a_N$  in the unit disk  $\mathbb{D} \subset \mathbb{C}$  and  $N$  matrices  $A_1, \dots, A_N \in \Omega_n$  decide whether there is a holomorphic map  $\varphi \in \mathcal{O}(\mathbb{D}, \Omega_n)$  such that  $\varphi(a_j) = A_j$ ,  $1 \leq j \leq N$ . This problem has been studied by many authors; we refer the reader to [1, 2, 3, 4, 6, 7] and the references there.

The study of the spectral Nevanlinna–Pick problem in the case  $N = 2$  reduces to the computation of the Lempert function of the spectral ball. Recall that for a domain  $D \subset \mathbb{C}^m$  the Lempert function of the domain  $D$  is defined as follows:

$$l_D(z, w) := \inf \{|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \varphi(\alpha) = w\}, \quad z, w \in D.$$

The infinitesimal version of the above problem, the so-called spectral Carathéodory–Fejér problem is the following one: given  $N + 1$  matrices

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$A_0, \dots, A_N$  in  $\mathcal{M}_n$  decide whether there is a map  $\varphi \in \mathcal{O}(\mathbb{D}, \Omega_n)$  such that  $A_j = \varphi^{(j)}(0)$ ,  $0 \leq j \leq N$ . This problem has been studied in [13].

The study of the spectral Carathéodory–Fejer problem in the case  $N = 1$  reduces to the computation of the Kobayashi–Royden pseudometric of the spectral ball. Recall that for a domain  $D \subset \mathbb{C}^m$  the Kobayashi–Royden pseudometric is defined as follows:

$$\begin{aligned} \kappa_D(z; X) := \inf \{|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \\ \varphi(0) = z, \alpha \varphi'(0) = X\}, \quad z \in D, X \in \mathbb{C}^m. \end{aligned}$$

In this note we point out some of the non-stability phenomena of both spectral problems which complicate their study.

First note that if we replace each of the matrices in the spectral Nevanlinna–Pick problem by similar ones then we do not change its solution.<sup>1</sup> A natural reduction of the problems is then to associate to each matrix its spectrum, or, in order to deal with  $n$ -tuples of complex numbers, the coefficients of its characteristic polynomial

$$P_A(t) := \det(tI - A) = t^n + \sum_{j=1}^n (-1)^j \sigma_j(A) t^{n-j},$$

where  $I \in \mathcal{M}_n$  is the unit matrix,

$$\sigma_j(A) := \sigma_j(\lambda_1, \dots, \lambda_n) := \sum_{1 \leq k_1 < \dots < k_j \leq n} \lambda_{k_1} \dots, \lambda_{k_j}$$

and  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$ .

Put  $\sigma = (\sigma_1, \dots, \sigma_n)$ . We shall consider  $\sigma$  as a map either from  $\mathcal{O}(\mathcal{M}_n, \mathbb{C}^n)$ , or from  $\mathcal{O}(\mathbb{C}^n, \mathbb{C}^n)$ . The set

$$\mathbb{G}_n := \{\sigma(A) : A \in \Omega_n\}$$

is called the symmetrized  $n$ -disk, and has been widely studied; we refer the reader to [3, 7, 15, 16, 17] and references there.

We recall a few definitions from linear algebra.

**Definition 1.** *Given a matrix  $A \in \mathcal{M}_n$ , the commutant of  $A$  is*

$$\mathcal{C}(A) := \{M \in \mathcal{M}_n : MA = AM\},$$

and the set of polynomials in  $A$ ,  $\mathcal{P}(A) \subset \mathcal{C}(A)$  is given by

$$\mathcal{P}(A) := \{M \in \mathcal{M}_n : M = p(A), \text{ for some } p \in \mathbb{C}[X]\}.$$

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<sup>1</sup>Indeed, assume that  $\varphi \in \mathcal{O}(\mathbb{D}, \Omega_n)$ ,  $\varphi(a_j) = A_j$  and  $A_j \sim \tilde{A}_j$ ,  $1 \leq j \leq N$ . Then  $\tilde{A}_j = e^{\hat{A}_j} A_j e^{-\hat{A}_j}$  for some  $\hat{A}_j \in \mathcal{M}_n$ . In the standard way we may find  $\hat{\varphi} \in \mathcal{O}(\mathbb{D}, \mathcal{M}_n)$  with  $\hat{\varphi}(a_j) = \hat{A}_j$ ,  $1 \leq j \leq N$ . Then  $\tilde{\varphi} := e^{\hat{\varphi}} \cdot \varphi \cdot e^{-\hat{\varphi}} \in \mathcal{O}(\mathbb{D}, \Omega_n)$  and  $\tilde{\varphi}(a_j) = A_j$ .

**Definition 2.** Given  $(a_0, \dots, a_{n-1}) \in \mathbb{C}^n$ , the associated companion matrix is

$$\begin{pmatrix} 0 & & -a_0 \\ 1 & 0 & \vdots \\ & \ddots & \vdots \\ & \ddots & 0 & -a_{n-2} \\ & & 1 & -a_{n-1} \end{pmatrix}.$$

The companion matrix associated to a matrix  $A$  is the one associated to the coefficients of its characteristic polynomial, namely we set  $a_j = (-1)^{n-j} \sigma_{n-j}(A)$ ,  $0 \leq j \leq n-1$ .

**Proposition 3.** A matrix  $A \in \mathcal{M}_n$  with the following equivalent properties is called non-derogatory.

- (1)  $A$  is similar to its companion matrix.
- (2) There exists a cyclic vector for  $A$ .
- (3) The characteristic and minimal polynomials of  $A$  coincide.
- (4) Different blocks in the Jordan normal form of  $A$  correspond to different eigenvalues (that is, each eigenspace is of dimension exactly 1).
- (5)  $\mathcal{C}(A) = \mathcal{P}(A)$ .
- (6)  $\text{rank}(\sigma_{*,A}) = n$ .
- (7)  $\dim \mathcal{C}(A) = n$ .
- (8) If  $\Phi_A : \mathcal{M}_n^{-1} \longrightarrow \mathcal{M}_n$ , where  $\mathcal{M}_n^{-1}$  stands for the set of invertible matrices, is defined by  $\Phi_A(P) := P^{-1}AP$ , then  $\text{rank}((\Phi_A)_{*,I_n}) = n^2 - n$  (its maximal possible value).

Most of those properties can be found in [11, pp. 135–147]; more precise references and (easy) complements are given in Section 4.

Recall that the  $j$ -th coordinate of  $\sigma_{*,A}(B)$  is the sum of all  $j \times j$  determinants obtained by taking a principal  $j \times j$  submatrix of  $A$  and replacing one column by the corresponding entries of  $B$ . In particular, the first coordinate of  $\sigma_{*,A}(B)$  equals  $\text{tr}(B)$ .

Denote by  $\mathcal{C}_n$  the set of all non-derogatory matrices in  $\Omega_n$ . Obviously  $\mathcal{C}_n$  is an open and dense subset of  $\Omega_n$ .

Note that if  $A_1, \dots, A_N$  belong to  $\mathcal{C}_n$ , then any mapping  $\varphi \in \mathcal{O}(D, \mathbb{G}_n)$  with  $\varphi(\alpha_j) = \sigma(A_j)$  can be lifted to a mapping  $\tilde{\varphi} \in \mathcal{O}(\mathbb{D}, \Omega_n)$  with  $\tilde{\varphi}(\alpha_j) = A_j$ ,  $1 \leq j \leq N$  (see [1]). This means that in a generic case the spectral Nevanlinna–Pick problem for  $\Omega_n$  with dimension  $n^2$  can be reduced to the standard Nevanlinna–Pick problem for  $\mathbb{G}_n$  with dimension  $n$ .

As a consequence of the existence of the lifting above, we have the equality

$$(1) \quad l_{\Omega_n}(A, B) = l_{\mathbb{G}_n}(\sigma(A), \sigma(B)), \quad A, B \in \mathcal{C}_n.$$

Note that  $\mathbb{G}_n$  is a taut domain (cf. [9], [15]). In particular, there always exist extremal discs for  $l_{\mathbb{G}_n}$  and  $l_{\mathbb{G}_n}$  is a continuous function. Thus the spectral Nevanlinna–Pick problem with data  $(\alpha_1, A_1), (\alpha_2, A_2) \in \mathbb{D} \times \mathcal{C}_n$  is solvable if and only if

$$l_{\mathbb{G}_n}(\sigma(A), \sigma(B)) \leq m(\alpha_1, \alpha_2) := \left| \frac{\alpha_1 - \alpha_2}{1 - \alpha_1 \bar{\alpha}_2} \right|.$$

An explicit formula for  $l_{\mathbb{G}_2}$  is found in [3]. The proof there is based on studying the complex geodesics of  $\mathbb{G}_2$ . It turns out that  $\tanh^{-1} l_{\mathbb{G}_2}$  coincides with the Carathéodory distance of  $\mathbb{G}_2$ . On the other hand,  $\mathbb{G}_2$  cannot be exhausted by domains biholomorphic to convex domains (see [5], [8]). So  $\mathbb{G}_2$  serves as the first counterexample to converse of the Lempert theorem (cf. [15]). In spite of this phenomenon,  $\tanh^{-1} l_{\mathbb{G}_n}$ , does not even satisfy the triangle inequality for  $n > 2$ , that is, it does not coincide with the Kobayashi distance of  $\mathbb{G}_n$  for  $n > 2$  (see [17]).

The behavior of  $l_{\Omega_n}$  is much more complicated when one of the arguments is derogatory. However, if  $A$  is a scalar matrix, say  $A = tI$ ,  $t \in \mathbb{D}$ , then (cf. [1])

$$(2) \quad l_{\Omega_n}(tI, B) = \max_{\lambda \in sp(B)} m(t, \lambda).$$

To prove (2), observe first that  $B \rightarrow (B - tI)(I - \bar{t}B)^{-1}$  is an automorphism of  $\Omega_n$ . So we may assume that  $t = 0$ . Then it remains to make use of the fact that  $l_{\Omega_n}(0, B)$  equals the Minkowski function of the balanced domain  $\Omega_n$  at  $B$ , that is,  $r(B)$ .

Since the  $2 \times 2$  derogatory matrices are scalar, we also get that the function  $l_{\Omega_2}$  is not continuous at the point  $(A, B)$  if and only if one of the matrices is scalar, say  $A$ , and the other one has two distinct eigenvalues (see [6]). In this case even  $l_{\Omega_2}(\cdot, B)$  is not continuous at  $A$  (but  $l_{\Omega_2}(A, \cdot)$  is continuous at  $B$ ). We shall show that this phenomenon extends to  $\Omega_n$ .

**Proposition 4.** *For  $B \in \mathcal{C}_n$  and  $t \in \mathbb{D}$  the following conditions are equivalent:*

- (i) *the eigenvalues of  $B$  are equal;*
- (ii) *the function  $l_{\Omega_n}$  is continuous at the point  $(tI, B)$ ;*
- (iii) *the function  $l_{\Omega_n}(\cdot, B)$  is continuous at the point  $tI$ ;*

At the infinitesimal level of the Kobayashi–Royden pseudometric, for  $A \in \mathcal{C}_n$  and  $B \in \mathcal{M}_n$  one has that (see Theorem 2.1 in [13])

$$(3) \quad \kappa_{\Omega_n}(A; B) = \kappa_{\mathbb{G}_n}(\sigma(A); \sigma_{*,A}(B)).$$

Since  $\kappa_{\Omega_n}(A; B) \geq \kappa_{\mathbb{G}_n}(\sigma(A); \sigma_{*,A}(B))$  for  $(A; B) \in \Omega_n \times \mathcal{M}_n$ ,  $\kappa_{\Omega_n}$  is an upper semicontinuous function and  $\kappa_{\mathbb{G}_n}$  is continuous (because  $\mathbb{G}_n$  is a taut domain), we get that  $\kappa_{\Omega_n}$  is a continuous function at any point  $(A; B) \in \mathcal{C}_n \times \mathcal{M}_n$ .

The things are more complicated if  $A \notin \mathcal{C}_n$ .

**Proposition 5.** *For  $B \in \mathcal{M}_n$  and  $t \in \mathbb{D}$  the following conditions are equivalent:*

- (i) *the eigenvalues of  $B$  are equal;*
- (ii) *the function  $\kappa_{\Omega_n}$  is continuous at the point  $(tI; B)$ ;*
- (iii) *the function  $\kappa_{\Omega_n}(\cdot; B)$  is continuous at the point  $tI$ ;*

Note that, similarly to the equality (2), one has that

$$(4) \quad \kappa_{\Omega_n}(tI; B) = \frac{\max_{\lambda \in sp(B)} |\lambda|}{1 - |t|^2}.$$

As a consequence of our considerations, we may also identify in a simple way the convex hull  $\hat{\Omega}_n$  of  $\Omega_n$ .

**Proposition 6.**  $\hat{\Omega}_n = \{A \in \mathcal{M}_n : |tr(A)| < n\}$ .

We now turn to analyzing the failure of hyperbolicity of  $\Omega_n$ . Observe first that if  $sp(A) \neq sp(B)$  then  $\sigma(A) \neq \sigma(B)$  and hence

$$l_{\Omega_n}(A, B) \geq l_{\mathbb{G}_n}(\sigma(A), \sigma(B)) > 0.$$

Then as a consequence of the proof of Lemma 13 in [9] we have the following

**Proposition 7.** *For any  $A, B \in \Omega_n$  the equality  $l_{\Omega_n}(A, B) = 0$  holds if and only if  $sp(A) = sp(B)$ . Moreover, in this case there is a  $\varphi \in \mathcal{O}(\mathbb{C}, \Omega_n)$  with  $\varphi(0) = A$ ,  $\varphi(1) = B$  and  $sp(\varphi(\lambda)) = sp(A)$  for any  $\lambda \in \mathbb{C}$ .*

It is natural to consider the infinitesimal version of this proposition.

First, note that the equality (4) implies that if  $A \in \Omega_n$  is a scalar matrix and  $B \in \mathcal{M}_n$ , then  $\kappa_{\Omega_n}(A; B) = 0$  if and only if  $sp(B) = 0$ . Conversely, if  $A$  is scalar and  $sp(B) = 0$ , then the linear mapping  $p : \lambda \rightarrow A + \lambda B$  has the following properties:  $p(0) = A$ ,  $p'(0) = B$  and  $sp(p(\lambda)) = sp(A)$  for any  $\lambda \in \mathbb{C}$ .

On the other hand, the equality (3) implies that if  $A \in \mathcal{C}_n$  and  $B \in \mathcal{M}_n$ , then  $\kappa_{\Omega_n}(A; B) = 0$  if and only if  $\sigma_{*,A}(B) = 0$ . Moreover, the following is true.

**Proposition 8.** *If  $A \in \mathcal{C}_n$ ,  $B \in \mathcal{M}_n$  and  $\kappa_{\Omega_n}(A; B) = 0$ , then there is a mapping  $\varphi \in \mathcal{O}(\mathbb{C}, \Omega_n)$  with  $\varphi(0) = A$ ,  $\varphi'(0) = B$  and  $sp(\varphi(\lambda)) = sp(A)$  for any  $\lambda \in \mathbb{C}$ .*

These observations let us state the following

**Conjecture 9.** *If  $A \in \Omega_n$ ,  $B \in \mathcal{M}_n$  and  $\kappa_{\Omega_n}(A; B) = 0$ , then there is a polynomial mapping  $p : \mathbb{C} \rightarrow \Omega_n$  of degree at most  $n$  with  $p(0) = A$ ,  $p'(0) = B$  and  $sp(p(\lambda)) = sp(A)$  for any  $\lambda \in \mathbb{C}$ .*

To support this conjecture, we shall prove it for  $n = 2$ .

The rest of the paper is organized as follows. The proof of Proposition 4 is given in Section 4. Section 5 contains the proofs of Propositions 5, 6 and 8, as well as Conjecture 9 for  $n = 2$ . The proof of Proposition 3 is discussed in Section 4.

## 2. PROOF OF PROPOSITION 4

We shall need the following

**Proposition 10.** (i) *If  $A, B \in \Omega_n$ , then*

$$l_{\mathbb{G}_n}(\sigma(A), \sigma(B)) \leq l_{\Omega_n}(A, B) \leq \min_{\pi} \max_{1 \leq j \leq n} m(\lambda_j, \mu_{\pi(j)}),$$

where  $sp(A) = \{\lambda_1, \dots, \lambda_n\}$ ,  $sp(B) = \{\mu_1, \dots, \mu_n\}$ , and the minimum is taken over all permutations  $\pi$  of  $\{1, \dots, n\}$ .

(ii) (see Theorem 5.2 in [7]) *If the eigenvalues of  $B(z) \in \mathcal{O}(\mathbb{D}, \Omega_n)$  have the form  $e^{i\theta \frac{1-\bar{\alpha}_j}{1-\alpha_j} \frac{z-\alpha_j}{1-z\bar{\alpha}_j}}$ , then*

$$l_{\mathbb{G}_n}(\sigma(B(z)), \sigma(B(w))) = l_{\Omega_n}(B(z), B(w)) = m(z, w).$$

(iii) *If  $B \in \Omega_n$  and  $t \in \mathbb{D}$ , then the eigenvalues of  $B$  are equal if and only if*

$$l_{\mathbb{G}_n}(\sigma(tI), \sigma(B)) = \max_{\lambda \in sp(B)} m(t, \lambda).$$

**Remark.** One may conjecture that Proposition 10 (ii) describes all the possibilities for the equality

$$l_{\mathbb{G}_n}(\sigma(A), \sigma(B)) = \min_{\lambda \in sp(A)} \max_{\mu \in sp(B)} m(\lambda, \mu).$$

Assuming Proposition 10 (iii), we are ready to prove Proposition 4.

The implication (ii)  $\Rightarrow$  (iii) is trivial. For the rest of the proof we may assume that  $t = 0$ .

We shall show that (i)  $\Rightarrow$  (ii) for any  $B \in \Omega_n$ . Let  $(A_j) \rightarrow 0$  and  $(B_j) \rightarrow B$ . Then, by Proposition 10 (iii) and (2),

$$l_{\Omega_n}(A_j, B_j) \geq l_{\mathbb{G}_n}(\sigma(A_j), \sigma(B_j)) \rightarrow l_{\mathbb{G}_n}(0, \sigma(B)) = r(B) = l_{\Omega_n}(0, B).$$

Thus the function  $l_{\Omega_n}(\cdot, B)$  is lower semicontinuous at the point  $(0, B)$ . Since it is (always) upper semicontinuous, we conclude that it is continuous at this point.

It remains to prove that  $(iii) \Rightarrow (i)$ . Since  $\mathcal{C}_n$  is a dense subset in  $\Omega_n$ , we may find  $\mathcal{C}_n \supset (A_j) \rightarrow 0$ . Then, by (2) and (1),

$$r(B) = l_{\Omega_n}(0, B) \leftarrow l_{\Omega_n}(A_j, B) = l_{\mathbb{G}_n}(\sigma(A_j), \sigma(B)) \rightarrow l_{\mathbb{G}_n}(0, \sigma(B))$$

and hence  $l_{\mathbb{G}_n}(0, \sigma(B)) = r(B)$ . It follows by Proposition 10 (iii) that the eigenvalues of  $B$  are equal.

This completes the proof of Proposition 4.

**Proof of Proposition 10 (i).** The first inequality is trivial.

To prove the second one recall that (see the footnote on page 2)

$$l_{\Omega_n}(A, B) = l_{\Omega_n}(A', B'), \quad A' \sim A, \quad B' \sim B.$$

So we may assume  $A = (a_{jk})$  and  $B = (b_{jk})$  are Jordan matrices with

$$\max_{1 \leq j \leq n} m(a_{jj}, b_{jj}) = s := \min_{\pi} \max_{1 \leq j \leq n} m(\lambda_j, \mu_{\pi(j)}).$$

Let  $s_1 > s$ . Then we may choose  $\varphi_{jj} \in \mathcal{O}(\overline{\mathbb{D}}, \mathbb{D})$  such that  $\varphi_{jj}(0) = a_{jj}$  and  $\varphi_{jj}(s_1) = b_{jj}$ . For  $\zeta \in \mathbb{C}$  set

$$\varphi_{jk}(\zeta) = \begin{cases} 0, & j > k \\ a_{jk} + \frac{b_{jk} - a_{jk}}{s_1} \zeta, & j < k. \end{cases}$$

Now  $\varphi = (\varphi_{jk}) \in \mathcal{O}(\overline{\mathbb{D}}, \Omega_n)$  which shows that  $l_{\Omega_n}(A, B) < s_1$ . Since  $s_1 > s$  was arbitrary, we are done.

**Remark.** Obvious modifications in the above proof imply Proposition 7.

**Proof of Proposition 10 (iii).** If the eigenvalues of  $B$  are equal, say to  $\lambda$ , then

$$l_{\mathbb{G}_n}(\sigma(tI), \sigma(B)) = m(t, \lambda)$$

by Proposition 10 (ii) with  $\alpha_j = 0$  (or, directly, considering the mapping  $\zeta \rightarrow \sigma(\zeta, \dots, \zeta)$  shows that second inequality in Proposition 10 (i) becomes equality).

To prove the converse, we shall need the following

**Lemma 11.** *Let  $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{T} = \partial\mathbb{D}$  be pairwise different points. Then for any  $\lambda_1, \dots, \lambda_n \in \mathbb{D}$ , there are  $\beta \in \mathbb{D}$  and a Blaschke product  $\mathcal{B}$  of order  $\leq n$  with  $\mathcal{B}(0) = 0, \mathcal{B}(\varepsilon_1\beta) = \lambda_1, \dots, \mathcal{B}(\varepsilon_n\beta) = \lambda_n$ .*

Assuming Lemma 11, we shall complete the proof of Proposition 10 (iii). We may assume that  $t = 0$ . Let  $\lambda_1, \dots, \lambda_n$  be the eigenvalues

of  $B$ . Set  $\sqrt[n]{1} = \{\varepsilon_1, \dots, \varepsilon_n\}$ . Let  $\beta \in \mathbb{D}$  and  $\mathcal{B}$  be as in Lemma 11. Consider the mapping

$$\zeta \rightarrow f_{\mathcal{B}}(\zeta) := \sigma(\mathcal{B}(\varepsilon_1 \sqrt[n]{\zeta}), \dots, \mathcal{B}(\varepsilon_n \sqrt[n]{\zeta}))$$

(where  $\sqrt[n]{\zeta}$  is arbitrary chosen). It is easy to see that  $f_{\mathcal{B}} \in \mathcal{O}(\mathbb{D}, \mathbb{G}_n)$ . Hence  $l_{\mathbb{G}_n}(0, \sigma(B)) \leq |\beta|^n$ . It remains to prove that if  $|\beta|^n \geq \max_{1 \leq j \leq n} |\lambda_j|$  then  $\lambda_1 = \dots = \lambda_n$ . We may assume that

$$\mathcal{B}(\zeta) = z \frac{a_0 \zeta^k + a_1 \zeta^{k-1} + \dots + a_k}{\bar{a}_k \zeta^k + \bar{a}_{k-1} \zeta^{k-1} + \dots + \bar{a}_0},$$

where  $a_0 = 1$  and  $k \leq n - 1$ . Then for  $1 \leq j \leq n$  one has that

$$|\beta|^{n-1} |\overline{a_k}(\varepsilon_j \beta)^k + \overline{a_{k-1}}(\varepsilon_j \beta)^{k-1} + \dots + \overline{a_0}| \geq |a_0(\varepsilon_j \beta)^k + a_1(\varepsilon_j \beta)^{k-1} + \dots + a_k|.$$

Squaring both sides of this inequality, we get that

$$\begin{aligned} |\beta|^{2n-2} \left( \sum_{s=0}^k |a_s|^2 |\beta|^{2s} + 2\Re \sum_{0 \leq p < s \leq k} a_p \overline{a_s} \beta^s \overline{\beta}^p \varepsilon_j^{s-p} \right) &\geq \\ \sum_{s=0}^k |a_s|^2 |\beta|^{2(k-s)} + 2\Re \sum_{0 \leq p < s \leq k} a_p \overline{a_s} \beta^{k-p} \overline{\beta}^{k-s} \varepsilon_j^{s-p}. \end{aligned}$$

Summing these inequalities for  $j = 1, \dots, n$  we get that

$$|\beta|^{2n-2} \sum_{s=0}^k |a_s|^2 |\beta|^{2s} \geq \sum_{s=0}^k |a_s|^2 |\beta|^{2(k-s)},$$

that is,

$$\sum_{s=0}^k |a_s|^2 (|\beta|^{2(n+s-1)} - |\beta|^{2(k-s)}) \geq 0.$$

Since  $k \leq n - 1$ , then  $k - s < n + s - 1$  if  $s > 0$  and hence  $a_s = 0$ . On the other hand,  $a_0 \neq 0$  and thus  $k = n - 1$ . It follows that  $\mathcal{B}(z) = z^n$  and then  $\lambda_1 = \dots = \lambda_n$ .

**Proof of Lemma 11.** Let  $S$  be the set of all  $\beta \in \mathbb{D}$  for which the Nevanlinna–Pick problem with data  $(0, 0), (\varepsilon_1 \beta, \lambda_1), \dots, (\varepsilon_n \beta, \lambda_n)$  is solvable. We may assume that  $0 \notin S$ ; otherwise, the identity mapping does the job. Then we must have  $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$  with  $f(\varepsilon_j \beta) = \lambda'_j := \frac{\lambda_j}{\varepsilon_j \beta}$ ,  $1 \leq j \leq n$ . The existence of such a function is equivalent to the semi-positivity of the matrix  $A(\beta) = [a_{j,k}(\beta)]_{j,k=1}^n$ , where  $a_{j,k}(\beta) = \frac{1 - \lambda'_j \bar{\lambda}'_k}{1 - \varepsilon_j \bar{\varepsilon}_k |\beta|^2}$ . Observe that  $a_{j,k}(\cdot)$ ,  $j \neq k$ , is bounded on  $\mathbb{D}$ . On the other hand,  $\lim_{\beta \rightarrow \mathbb{T}} a_{j,j}(\beta) = +\infty$ . Thus the matrix  $A(\beta)$  is

(strictly) positive for  $\beta$  near  $\mathbb{T}$ . Since  $0 \notin S$ , it follows that  $S$  is a proper non-empty (circular) closed subset of  $\mathbb{D}$ . So there is a boundary point  $\beta_0 \in \mathbb{D}$  of  $S$ . Then  $A(\beta_0)$  is not strictly positive which means that  $m = \text{rank}(A(\beta_0))$  is not maximal, that is,  $m < n$ . This implies that the respective Nevanlinna–Pick problem has a unique solution and it is a Blaschke product  $\tilde{\mathcal{B}}$  of order  $m$  (cf. [10]). It remains to set  $\mathcal{B}(\zeta) = \zeta \tilde{\mathcal{B}}(\zeta)$ .

### 3. PROOFS OF PROPOSITIONS 5, 6, 8 AND CONJECTURE 9 FOR $n = 2$ .

**Proof of Proposition 5.** The implication  $(ii) \Rightarrow (iii)$  is trivial. For the rest of the proof we may suppose as above that  $t = 0$ .

We shall show that  $(i) \Rightarrow (ii)$  for any  $B \in \Omega_n$ . Let  $(A_j) \rightarrow 0$  and  $(B_j) \rightarrow B$ . Then, by (4) and the equality  $\kappa_{\mathbb{G}_n}(0; e_1) = \frac{1}{n}$  (cf. [17]),

$$\begin{aligned} \kappa_{\Omega_n}(A_j; B_j) &\geq \kappa_{\mathbb{G}_n}(\sigma(A_j); \sigma_{*,A_j}(B_j)) \rightarrow \kappa_{\mathbb{G}_n}(0; \sigma_{*,0}(B)) = \\ \kappa_{\mathbb{G}_n}(0; \text{tr}(B)e_1) &= \frac{|\text{tr}(B)|}{n} = r(B) = \kappa_{\Omega_n}(0; B). \end{aligned}$$

Thus, the function  $\kappa_{\Omega_n}$  is lower semicontinuous at the point  $(0; B)$ . Since it is (always) upper semicontinuous, we conclude that it is continuous at this point.

It remains to prove that  $(iii) \Rightarrow (i)$ . Since  $\mathcal{C}_n$  is a dense subset of  $\Omega_n$ , we may find  $\mathcal{C}_n \supset (A_j) \rightarrow 0$ . Then, by (4) and (3),

$$\begin{aligned} r(B) &= \kappa_{\Omega_n}(0; B) \leftarrow \kappa_{\Omega_n}(A_j; B) = \\ \kappa_{\mathbb{G}_n}(\sigma(A_j); \sigma_{*,A_j}(B)) &\rightarrow \kappa_{\mathbb{G}_n}(0; \sigma_{*,0}(B)) = \frac{|\text{tr}(B)|}{n}. \end{aligned}$$

Hence  $r(B) = \frac{|\text{tr}(B)|}{n}$ , that is, the eigenvalues of  $B$  are equal.

**Proof of Proposition 6.** Since  $\Omega_n$  is a balanced domain, we have that (cf. [15])

$$h_{\hat{\Omega}_n} = k_{\Omega_n}(0, \cdot),$$

where  $h_{\hat{\Omega}_n}$  and  $k_{\Omega_n}$  are the Minkowski function of  $\hat{\Omega}_n$  and the Kobayashi distance of  $\Omega_n$ , respectively.

On the other hand, since  $k_{\Omega_n}$  is a continuous function, the density of  $\mathcal{C}_n$  in  $\Omega_n$  and the equality (1) imply that

$$k_{\Omega_n}(A, B) = k_{\mathbb{G}_n}(\sigma(A), \sigma(B)), \quad A, B \in \Omega_n.$$

It follows that for any  $t \in \mathbb{D} \setminus \{0\}$

$$h_{\hat{\Omega}_n}(A) = \frac{k_{\Omega_n}(0, tA)}{|t|} =$$

$$\frac{k_{\mathbb{G}_n}(0, \sigma(tA))}{|t|} = \frac{k_{\mathbb{G}_n}(0, t \cdot \text{tr}(A)e_1 + o(t))}{|t|}.$$

Denote by  $\hat{\kappa}_{\mathbb{G}_n}(0; \cdot)$  the Kobayashi–Buseman metric of  $\mathbb{G}_n$  at 0, that is, the largest norm bounded above by  $\kappa_{\mathbb{G}_n}(0; \cdot)$ . Since  $\mathbb{G}_n$  is a taut domain, we have that (see [18])

$$\lim_{t \rightarrow 0} \frac{k_{\mathbb{G}_n}(0, t \cdot \text{tr}(A)e_1 + o(t))}{|t|} = |\text{tr}(A)|\hat{\kappa}_{\mathbb{G}_n}(0; e_1).$$

Making use of the equality  $\hat{\kappa}_{\mathbb{G}_n}(0; e_1) = \frac{1}{n}$  (cf. [17]), we get that

$$\hat{\Omega}_n = \{A \in \mathcal{M}_n : h_{\hat{\Omega}_n}(A) = \frac{|\text{tr}(A)|}{n} < 1\}.$$

**Remark.** An algebraic approach in the proof of Proposition 6 also works.

**Proof of Proposition 8.** By (3), the equality  $\kappa_{\Omega_n}(A; B)$  is equivalent to  $\sigma_{*,A}(B) = 0$ . By property (8) in Proposition 3 and its proof, we have a matrix  $Y \in \mathcal{M}_n$  such that  $-YA + AY = B$ . Then the mapping  $\lambda \rightarrow e^{-\lambda Y}Ae^{\lambda Y}$  satisfies all the required properties.

**Proof of Conjecture 9 for  $n=2$ .** If  $A$  is derogatory, it is scalar. Then  $\text{sp}(B) = 0$  and so the linear mapping  $\lambda \rightarrow A + \lambda B$  does the job.

Let  $A$  be non-derogatory. Choose  $r > 0$  and  $\varphi \in \mathcal{O}(r\mathbb{D}, \Omega_n)$  such that  $\varphi(0) = A$ ,  $\varphi'(0) = B$  and  $\text{sp}(\varphi(\lambda)) = \text{sp}(A)$  for any  $\lambda \in r\mathbb{D}$ . Then

$$\varphi(\lambda) = A + \lambda B + \lambda^2\psi(\lambda), \quad \psi \in \mathcal{O}(r\mathbb{D}, \mathcal{M}_n).$$

The Taylor expansion shows that the condition  $\text{sp}(\varphi) = \text{sp}(A)$ , that is

$$\text{tr}(\varphi) = \text{tr}(A) \text{ and } \det \varphi = \det A,$$

is equivalent to  $\text{tr}(B) = \text{tr}(\psi) = 0$  and

$$f(A, B) = \det B + f(A, \psi) = f(B, \psi) = \det \psi = 0,$$

where

$$f(C, D) = c_{11}d_{22} + c_{22}d_{11} - c_{12}d_{21} - c_{21}d_{12}, \quad C, D \in \mathcal{M}_2$$

Observe that the quadratic mapping  $\lambda \rightarrow A + \lambda B + \lambda^2\psi(0)$  satisfies the same conditions. Therefore it has the desired properties.

#### 4. APPENDIX: PROOF OF PROPOSITION 3

In [11, definition 3.2.4.1, p. 135], property (4) is taken as the definition of *nondiagonalizability*. The fact that (4) implies (5) is [11, Theorem 3.2.4.2, p. 135]. The converse implication is stated in [11, p. 137], and proved in [12, Corollary 4.4.18, p. 275]. The fact that (7) is equivalent to (4) is part of [12, Theorem 4.4.17, p. 275]. The equivalence between

(1) and (3) is [11, Theorem 3.3.15, p. 147]. The equivalence of those properties with (4) is left as an exercise immediately after this.

From the form of a companion matrix, it is immediate that if we denote by  $e_1$  the first basis vector, its iterates  $e_1, Ae_1, \dots, A^{n-1}e_1$  generate  $\mathbb{C}^n$ , and similarity preserves this property. Conversely, if one has a cyclic vector, it is easy to see that the space is actually generated by the first  $n$  iterates as above, and that they must form a basis, in which the matrix will take the companion form. So (2) is equivalent to (1).

We now move on to the statements about ranks. First note that

$$\Phi_A(I + H) := (I + H)^{-1}A(I + H) = A + (-HA + AH) + O(H^2),$$

so  $\dim \text{Ker}((\Phi_A)_{*,I_n}) = \dim \mathcal{C}(A)$ , and, by the rank theorem, (8) is equivalent to (7). The comment about maximality follows from [12, Theorem 4.4.17(d), p. 275].

To study  $\sigma_{*,A}$ , first note that for  $P \in \mathcal{M}_n^{-1}$ ,

$$\sigma_{*,A}(H) = \sigma_{*,P^{-1}AP}(P^{-1}HP),$$

so  $\text{rank}(\sigma_{*,A})$  is preserved when we pass to a similar matrix. Thus, if  $A$  verifies (1), we may suppose then that it is a companion matrix. Choose  $H = (h_{i,j})$  such that  $h_{i,j} = 0$ ,  $1 \leq j \leq n - 1$ . Then  $A + H$  is also a companion matrix, and  $\sigma$  when restricted to that set is a linear map in the last column. Then the mapping

$$\sigma_{*,A}(H) = (-h_{n,n}, h_{n-1,n}, \dots, (-1)^{n-1}h_{1,n}),$$

is onto  $\mathbb{C}^n$ . So (6) follows by (1).

To complete the proof of Proposition 3, it is enough to show (6) implies (4).

Given any  $\lambda \in \mathbb{C}$ , let  $A_\lambda := A - \lambda I_n$ . Then  $P_A(X) = P_{A_\lambda}(X + \lambda)$ , so that  $\sigma(A)$  is a polynomial expression (involving the parameter  $\lambda$ ) of the components of  $\sigma(A_\lambda)$ . Therefore  $\text{rank}(\sigma_{*,A}) \leq \text{rank}(\sigma_{*,A_\lambda})$ .

Suppose now that property (6) holds and (4) does not. Let  $\lambda$  be an eigenvalue such that  $\dim \text{Ker}(A - \lambda I_n) \geq 2$ . Choose a basis of  $\mathbb{C}^n$  containing a basis of  $\text{Ker}(A - \lambda I_n)$ . In this basis, the matrix  $A - \lambda I_n$  transforms into a matrix with at least two columns which are identically zero, and therefore  $\sigma_n(A - \lambda I_n + H)$  is a polynomial containing only monomials of degree at least 2 in the  $h_{i,j}$ . This implies that  $(\sigma_n)_{*,A-\lambda I_n} = 0$  and therefore

$$\text{rank}(\sigma_{*,A}) \leq \text{rank}(\sigma_{*,A-\lambda I_n}) \leq n - 1,$$

which is a contradiction.

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